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Superposition Principle and Sectors in Quantum Logics

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The superposition principle and sectors are studied and their properties in some types of logic are derived.

1. INTRODUCTION

Let L be a logic, i.e. an orthocomplemented partially ordered set with the first and last elements 0 and 1, respectively, in which $\bigvee a_i \in L$ for any sequence $\{a_i\} \subset L$ such that $a_i \leqslant a_j^{\perp}$, $i \neq j, i,j = 1,2,\ldots$ and which has the orthomodularity property: $a \leqslant b (a,b \in L)$ implies that there is a $d \in L, d \leqslant a^{\perp}$ and such that $b = a \vee d$. The orthocomplementation $a \mapsto a^{\perp}$ in L has the following properties: (i) $(a^{\perp})^{\perp} = a$, (ii) $a \leqslant b \Leftrightarrow b^{\perp} \leqslant a^{\perp}$, (iii) $a \vee a^{\perp} = 1$. The elements $a,b \in L$ are disjoint (written $a \perp b$) if $a \leqslant b^{\perp}$. The elements $a,b \in L$ are compatible (written $a \leftrightarrow b$) if there are $a_1, b_1, c \in L$, mutually disjoint and such that $a = a_1 \vee c$ and $b = b_1 \vee c$. We shall also assume that if $a,b,c \in L$ are mutually compatible then $a \leftrightarrow b \vee c$. A map $m: L \to [0,1]$ which satisfies (1)$m(1) = 1$, (2)$m(\bigvee a_i) = \sum m(a_i)$ for any sequence $\{a_i\}$ of mutually disjoint elements of L, is a state on L. If m is a state that cannot be written in the form $m = cm_1 + (1-c)m_2$, where $0 < c < 1$ and m_1, m_2 are distinct states, then m is called a pure state. Let M be a set of states on L and let P be the set of all pure states of L contained in M. If $a \in L$, $m \in P$, define $P_a = \{m \in P : m(a) = 1\}$, $L_m = \{a \in L : m(a) = 1\}$. If $P_a \subset P_b$ implies $a \leqslant b (a,b \in L)$ and $L_{m_1} \subset L_{m_2}$ implies $m_1 = m_2$ $(m_1, m_2 \in P)$, we call (L,M) a quantum logic. Let (L,M) be a quantum logic; then for any $a \in L$, $a \neq 0$ there is an $m \in P$ such that $m(a) = 1$. Indeed, if not, then the relation $P_a = P_0$ implies $a = 0$, a contradiction. Let $L_m^0 = \{a \in L : m(a) = 0\}$. Clearly,

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$L_m=\{a^\perp : a\in L_m^0\}$ and $L_m^0=\{a^\perp : a\in L_m\}$. A state $m\in M$ is a superposition of the states $p,q\in M$ if $p(a)=0$ and $q(a)=0$ imply $m(a)=0$, or, equivalently, if $p(a)=1$ and $q(a)=1$ imply $m(a)=1$. A set $S\subset P$ is said to be closed under superpositions if it contains every pure superposition of any pair of its elements. If $S\subset P$ is not closed under superpositions, let $\Lambda(S)$ denote the smallest subset of P closed under superpositions and containing S. The set $S\subset P$ is a sector if (i) $S=\Lambda(S)$, (ii) if $p,q\in S$ then there is an $r\in\Lambda(\{p,q\})$ such that $r\neq p, r\neq q$, (iii) if $q\in P, q\notin S$ then $\Lambda(\{p,q\})=\{p,q\}$ for any $p\in S$. We say that the superposition principle holds in (L,M) if $\Lambda(\{p,q\})\neq\{p,q\}$ for any $p,q\in P, p\neq q$ (Pulmannová, 1976).

Let C be the set of all elements of L that are compatible with all the other elements, i.e., $C=\{a\in L: a\leftrightarrow b \text{ for all } b\in L\}$. C is called the center of L. It was shown (Varadarajan, 1962) that C is a Boolean sub-σ-algebra of L. A logic L is called irreducible if its center C is trivial, i.e., if $C=\{0,1\}$. If the superposition principle holds in a quantum logic (L,M), then L is irreducible (Pulmannová, 1976). If p is a pure state and $c\in C$, then $p(c)=1$ or $p(c)=0$ (Varadarajan, 1968).

The center C of a logic L is discrete if there exists an at most countable set $\{c_n\}_{n\in D}$ of mutually disjoint elements of C such that C consists precisely of all the lattice sums $\bigvee\{c_n : n\in Z\}$, where Z is an arbitrary subset of D. The c_n are called atoms of C. If we define $L_j=L[0,c_j]=\{b: b\in L, b\leqslant c_j\}$, then $L_j, j\in D$, are irreducible logics (c_j is the first element of L_j). The logic L can be thought of as a direct sum of the irreducible logics L_j. If p is a pure state of L_j, we define $\tilde{p}$ by $\tilde{p}(a)=p(a\wedge c_j)(a\in L)$. Then $\tilde{p}$ is a pure state on L. Varadarajan (1968) has shown that the set $\tilde{P}$ of all pure states of L can be written in the form $\tilde{P}=\cup\tilde{P}_j$, where $\tilde{P}_j=\{\tilde{p}: p \text{ is a pure state on } L_j\}$. To any state m on L we can find the states m_i on L_i such that $m(b)=\sum m_i(b\wedge c_i)m(c_i)$, $(b\in L)$ if we set $m_i(b\wedge c_i)=m(b\wedge c_i)/m(c_i)$, $m(c_i)\neq 0$. If $m(c_i)=0$, we set $m_i=0$. Let M_i be the set of m_i for all $m\in M$. If (L,M) is a quantum logic, then (L_i,M_i) are also quantum logics. If the irreducible quantum logics (L_i,M_i) are such that the superposition principle holds in them, then the sets $P_i=\tilde{P}_i\cap M$ are the sectors in P (Pulmannová, 1976).

2. CENTER OF A QUANTUM LOGIC AND SECTORS

Let (L,M) be a quantum logic and let C be the (nontrivial) center of L. For $s_1,s_2\in P$ we set $s_1\sim s_2$ if $s_1(c)=s_2(c)$ for all $c\in C$. Clearly, $\sim$ is a relation of equivalence. Let $[s]$ denote the equivalence class of P containing $s\in P$.

Theorem 1. If $S \subset P$ is a sector, then $p \sim q$ for any $p, q \in S$.

Proof. Let $p, q \in S$ be such that $p \not\sim q$. Then there is an element $c \in C$ such that $p(c) = 1$ (say) and $q(c) = 0$. By the proof of Theorem 3 in Pulmannová (1976) then $\Lambda(\{p, q\}) = \{p, q\}$, which contradicts the supposition that S is a sector. Hence $p \sim q$.

Theorem 2. Let (L, M) be a quantum logic such that the center C of L is discrete. Let L be the direct sum of $L_{[0, c_i]} = L_i$, where the quantum logics (L_i, M_i) are such that the superposition principle holds in them. Then for any $p, q \in P$, p and q belong to the same sector if and only if $p \sim q$.

Proof. Let $P_i = \{\tilde{p} \in P : p$ is a pure state on $L_i\}$. Then P_i are sectors in P. Let $c_i, i = 1, 2, \ldots$ be the atoms of C. Clearly, $p \sim q$ iff $p(c_i) = q(c_i)$ for all i. Let $p \sim q$ and let $p \in P_{i_0}$. Then $p(c_{i_0}) = q(c_{i_0}) = 1$ and $p(c_i) = q(c_i) = 0$ for $i \neq i_0$, which implies that $q \in P_{i_0}$, i.e., p and q belong to the same sector P_{i_0}. ■

An observable on the logic L is a map $x: B(R) \to L$ from the Borel subsets $B(R)$ of the real line R into L such that (i) $x(R) = 1, x(\emptyset) = 0$; (ii) if $E \cap F = \emptyset$ then $x(E) \perp x(F), E, F \in B(R)$; (iii) $x(\cup E_i) = \bigvee x(E_i)$, if $E_i \cap E_j = \emptyset$ for $i \neq j, i, j = 1, 2, \ldots$. The spectrum $\sigma(x)$ of an observable x is the smallest closed set $F \subset R$ such that $x(F) = 1$. An observable x is bounded if $\sigma(x)$ is bounded. The norm of a bounded observable x is defined by $\|x\| = \sup \{|t| : t \in \sigma(x)\}$. The expectation of an observable x in a state m is $m(x) = \int \lambda m(x(d\lambda))$, if the integral exists.

For a quantum logic (L, M) we define a metric on M by setting $d(p, q) = \sup \{|p(x) - q(x)| : x \in X_1\}$, where X_1 is the set of all observables x on L such that $\|x\| \leqslant 1$. It can be shown that $\|x\| = \sup \{|m(x)| : m \in M\}$ (Gudder, 1965). From this it follows that $d(p, q) \leqslant 2$ for any $p, q \in M$. An observable x on L is simple if $\sigma(x) \subset \{0, 1\}$. If $a \in L$, let x_a be the simple observable such that $x_a(\{1\}) = a$. For any state m then $m(x_a) = m(a)$.

Theorem 3. Let (L, M) be a quantum logic and let $p, q \in P$ be such that $d(p, q) < 2$. Then $p \sim q$.

Proof. Let $p \not\sim q$ and let $c \in C$ be such that $p(c) = 1, q(c) = 0$. Let us set $x = x_c - x_c{}^{\perp}$. [As x_c and $x_c{}^{\perp}$ are compatible, $x_c - x_c{}^{\perp}$ exists (Varadarajan, 1968)]. Let m be a state on L. Then $m(x) = m(x_c) - m(x_c{}^{\perp}) = 2m(c) - 1$; i.e., $|m(x)| \leqslant 1$, hence $\|x\| \leqslant 1$. Then $p(x) - q(x) = p(c) + q(c^{\perp}) = 2$, from which it follows that $d(p, q) = 2$. ■

Theorem 4. In a quantum logic (L,M), the equivalence classes $[s], s \in P$ are open–closed sets in the topology induced by the metric d.

Proof. Let p be an element of $[s]$. By Theorem 3, the set $\{q \in P: d(p,q) < 2\}$ is contained in $[s]$. This implies that $[s]$ is open. As the equivalence classes $[s], s \in P$, form a disjoint covering of P, they are also closed. ∎

If the conditions of Theorem 2 are satisfied, then the sectors are open–closed sets in P. Thus Theorem 5 can be considered as an analog of Proposition 4.5 in Roberts and Roepstorff (1969).

3. MINIMAL SUPERPOSITION PRINCIPLE AND SECTORS

Let (L,M) be a quantum logic and P be the set of all pure states contained in M. For $S \subset P$ and $a \in L$, let us write $S(a)=0$ if $s(a)=0$ for all $s \in S$. Let $\bar{S}=\{m \in P: S(a)=0 \text{ imply } m(a)=0\}$ (see Varadarajan, 1968; Gudder, 1970a, b; Berzi and Zecca, 1974). Gudder (1970b) introduced the following postulate (minimal superposition postulate, MSP): If S is any finite subset of P and $m \in \bar{S}$ is such that $m \notin \bar{Q}$ for any subset $Q \subsetneq S$, then $\{m, S_1\}^- \cap \bar{S}_2 \neq \varnothing$ for any $S_1, S_2 \subset P$ such that $S_1 \cap S_2 = 0$ and $S_1 \cup S_2 = S$.

Lemma 1. For any $S \subset P, \Lambda(S) \subset \bar{S}$.

Proof. Let $p, q \in \bar{S}$ and let $p(a)=0$, $q(a)=0$ imply $r(a)=0, r \in P, a \in L$. Then $S(a)=0$ imply $r(a)=0$, i.e., $r \in \bar{S}$. From this it follows that $\bar{S}$ is closed under superpositions. From $S \subset \bar{S}$ we get $\Lambda(S) \subset \Lambda(\bar{S}) = \bar{S}$. ∎

Lemma 2. If the MSP holds in (L,M) then $r \in \Lambda(\{p,q\})$ implies $q \in \Lambda(\{r,p\}), p \in \Lambda(\{r,q\})$ for any mutually different $p,q,r \in P$.

Proof. As $\Lambda(\{p,q\}) = \{p,q\}^-$ (Pulmannová, 1976), by the MSP we have that $\Lambda(\{r,p\}) \cap \{q\} \neq \varnothing$, i.e., $q \in \Lambda(\{r,p\})$. Analogically we get $p \in \Lambda(\{r,q\})$. ∎

Theorem 5. If the MSP holds in (L,M) then $\Lambda(S) = \bar{S}$ for any finite subset $S \subset P$.

Proof. For any $s,p \in P, \Lambda(\{s,p\}) = \{s,p\}^-$. We shall proceed by induction. Let $\Lambda(\{s_1, \ldots, s_k\}) = \{s_1, \ldots, s_k\}^-$ for any $k \leqslant n$ and for any $\{s_1, \ldots, s_k\} \subset P$. Let $s \in \{s_1, \ldots, s_{n+1}\}^-$ be such that $s \notin \bar{S}$ for any $S \subsetneq \{s_1, \ldots, s_{n+1}\}$. By the MSP then there is a $p \in P, p \in \{s, s_{n+1}\}^{-} \cap \{s_1, \ldots, s_n\}^{-} = \Lambda(\{s, s_{n+1}\}) \cap \Lambda(\{s_1, \ldots, s_n\})$. Then $p \in \Lambda(\{s_1, \ldots, s_n\}) \subset \Lambda(\{s_1, \ldots, s_{n+1}\})$. By Lemma 2, $s \in \Lambda(\{p, s_{n+1}\})$, so that $s \in \Lambda(\{s_1, \ldots, s_{n+1}\})$. Now let $q \in \{s_1, \ldots, s_{n+1}\}^{-}$ be such that there is a subset $S \subset \{s_1, \ldots, s_{n+1}\}$ such that $q \in \bar{S}$. Then $\bar{S} = \Lambda(S) \subset \Lambda(\{s_1, \ldots, s_{n+1}\})$ imply $q \in \Lambda(\{s_1, \ldots, s_{n+1}\})$. Thus we obtain $\{s_1, \ldots, s_{n+1}\}^{-} \subset \Lambda(\{s_1, \ldots, s_{n+1}\})$. The converse implication follows from Lemma 1. Hence, $\Lambda(S) = \bar{S}$ for any finite $S \subset P$. ■

For $p, q \in P, p \neq q$, let us set $p \approx q$ if there is an $r \in P$ such that $r \in \Lambda(p, q\}), r \neq p, r \neq q$. Let us set $p \approx p, p \in P$.

Theorem 6. Let the MSP hold in the quantum logic (L, M) and let $\Lambda(\{p, q, r\}) \neq \Lambda(\{p, q\}) \cup \Lambda(\{q, r\})$ for any distinct states $p, q, r \in P$ such that $p \approx q, q \approx r, r \notin \Lambda(\{p, q\})$. Then $\approx$ is an equivalence relation

Proof. We have only to show the transitivity. Let $p \approx q, q \approx r, p, q, r \in P$ be distinct states. If $r \in \Lambda(\{p, q\})$ then $q \in \Lambda(\{r, p\})$ by Lemma 2, which implies $r \approx p$. Let $r \notin \Lambda(\{p, q\})$ and let $m \in \Lambda(\{p, q, r\}), m \notin \Lambda(\{p, q\}), m \notin \Lambda(\{q, r\})$. Clearly, $m \neq p, q, r$. If $m \in \Lambda(\{p, r\})$ then $p \approx r$. If $m \notin \Lambda(\{p, r\})$, then we get by the MSP that $\Lambda(\{m, q\}) \cap \Lambda(\{p, r\}) \neq \varnothing$. Let $s \in \Lambda(\{m, q\}) \cap \Lambda(\{p, r\})$. If $s = p$ then $p \in \Lambda(\{m, q\})$ implies $m \in \Lambda(\{p, q\})$, a contradiction. Thus $s \neq p$ and analogically $s \neq r$. Hence $p \approx r$. ■

The following theorem shows that the condition in Theorem 6 is also necessary.

Theorem 7. Let the MSP hold in (L, M) and let $\approx$ be an equivalence relation. Then $p \approx q, q \approx r$ and $\Lambda(\{p, q, r\}) = \Lambda(\{p, q\}) \cup \Lambda(\{q, r\})$ imply $r \in \Lambda(\{p, q\})$ (p, q, r distinct states in P).

Proof. Let the MSP hold in (L, M) and let $\approx$ be an equivalence relation. Let $p \approx q$ and $q \approx r$. Then there is an $m \in \Lambda(\{p, r\}), m \neq p, m \neq r$. $\Lambda(\{p, r\}) \subset \Lambda(\{p, q, r\}) = \Lambda(\{p, q\}) \cup \Lambda(\{q, r\})$ implies $m \in \Lambda(\{p, q\})$ or $m \in \Lambda(\{q, r\})$. If $m \in \Lambda(\{p, q\}), m \neq q$, then $q \in \Lambda(\{m, p\})$. But $m \in \Lambda(\{p, r\})$ implies that $\Lambda(\{m, p\}) \subset \Lambda(\{p, r\})$, so that $q \in \Lambda(\{p, r\})$, i.e., $r \in \Lambda(\{p, q\})$. We get the same if $m = q$. If $m \in \Lambda(\{q, r\})$ we get analogically that $r \in \Lambda(\{p, q\})$. ■

Theorem 8. If the suppositions of Theorem 6 hold, then P can be written as the union of sectors.

Proof. We shall show that the equivalence classes P_λ of P by the relation $\approx$ are sectors. (i) Let $p,q\in P, p\approx q$ and let $r\in\Lambda(\{p,q\})$. If $r\neq p,q$ then $r\in\Lambda(\{p,q\})$ implies $p\in\Lambda(\{r,q\})$, i.e., $r\approx q$. From this it follows that $\Lambda(P_\lambda)=P_\lambda$. (ii) and (iii) from the definition of sectors are evident. ■

4. SUPERPOSITION PRINCIPLE IN PROJECTIVE LOGICS

An orthomodular σ lattice L is called a projective logic if the following conditions are satisfied (Varadarajan, 1968):

(i) Given $a\neq 0$ in L, there is a point $x\leqslant a$.
(ii) If $a\neq 0$ in L is the lattice sum of a finite set of points then $L_{[0,a]}=\{b\in L: b\leqslant a\}$ is a geometry of finite rank; we shall say that a is a finite element of L and write $\dim(a)$ for the dimension of $L_{[0,a]}$.
(iii) If $x,a\in L, a\neq 0,1$ and x is a point, then there are points $y,z\in L$ such that $y\leqslant a, z\leqslant a^\perp$ and $x\leqslant y\vee z$.
(iv) There exists at least one $a\in L$ such that $4\leqslant\dim(a)$.

If L is projective and its lattice is complete, then every element a of L is the lattice sum of the points it contains. A projective logic with the property that any family of mutually disjoint points of L is at most countable (i.e., L is separable) is complete (Varadarajan, 1968).

Let m be a state on a logic L. Let us set $L_m=\{b\in L: m(b)=1\}$. We shall say that m is supported if there is an element $a_m\in L, a_m\neq 0$ such that $L_m=\{b\in L: b\geqslant a_m\}$. The element a_m is called a support of the state m.

Theorem 9. Let (L,M) be a quantum logic such that L is a separable projective logic. In addition, let the following condition be satisfied: $m(a)=m(b)=0$ imply $m(a\vee b)=0 (a,b\in L, m\in P)$. Then the superposition principle holds on (L,M).

Proof. First we show that any $s\in P$ is supported (see also Zierler, 1961; Bugajska and Bugajski, 1972). If $L_s^0=\{a\in L: s(a)=0\}=\{0\}$ then $L_s=\{1\}$. But then $L_s\subset L_m$ for any $m\in P$, which implies $s=m$, which is impossible. From this it follows that $L_s^0\neq\{0\}$. Then there exists at least one maximal subset of mutually disjoint elements in L_s^0 (by the Zorn lemma) and this subset is at most countable. Let b be the supremum of this subset. Obviously b is a maximal element of L_s^0. Let $a\in L_s^0$, then $s(a)=0, s(b)=0$ imply $s(a\vee b)=0$. From the maximality of b it follows that $a\vee b=b$, i.e., $a\leqslant b$. Thus $L_s^0=\{a\in L: a\leqslant b\}$, i.e., $L_s=\{a\in L: b^\perp\leqslant a\}$, that is, $b^\perp$ is the support of s. We show that $b^\perp$ is a point. Let e be a point of L such that $e \leqslant b^\perp$. Then there is a state $q \in P$ such that $q(e)=1$. Let $a \in L_q$, then from $q(e)=1, q(a)=1$ it follows that $q(a \wedge e)=1$, i.e., $e \leqslant a$. Hence, $L_q=\{a \in L: e \leqslant a\}$. But then $L_s \subset L_q$ implies $s=q$, i.e., $b^\perp = e$. Now let $s_1, s_2 \in P$, $s_1 \neq s_2$. Then there are points e_1, e_2 such that $s_1(e_1)=1$, $s_2(e_2)=1$. As $L_{[0, e_1 \vee e_2]}$ is a geometry, there is a point $e_3 \in L, e_3 \neq e_1, e_2$ such that $e_3 \leqslant e_1 \vee e_2$. Let $s_3 \in P$ be such that $s_3(e_3)=1$. Let $a \in L$ be such that $s_1(a)=1, s_2(a)=1$. Then $e_1 \leqslant a, e_2 \leqslant a$ imply $e_3 \leqslant e_1 \vee e_2 \leqslant a$, i.e., $s_3(a)=1$. Hence, $s_3 \in \Lambda(\{s_1, s_2\})$. Clearly, $s_3 \neq s_1, s_2$. Indeed, if $s_3 = s_1$ then $s_3(e_3)=1, s_3(e_1)=1$ imply $s_3(e_1 \wedge e_3)=1$, so that $e_1 \wedge e_3 \neq 0$, i.e., $e_1 = e_3$, a contradiction. Hence, the superposition principle holds in (L, M). ■

We note that in the case of a general projective logic (not necessarily separable) the superposition principle need not hold unrestrictedly if there are states in P with no supports.

Lemma 3. Let (L, M) be a quantum logic such that L is a complete lattice and all states in P are supported. Let us denote by $\operatorname{supp} m$ the support of the state m. Then the following hold.

(i) If $m \in P$, then $\operatorname{supp} m$ is a point in L.

(ii) Given $x \in L, x \neq 0$, there is a point $a \in L$ such that $a \leqslant x$.

(iii) Every element $x \in L$ is the lattice sum of points contained in it.

(iv) $m(a_\alpha)=1, \alpha \in A, A$ is any set, then $m(\wedge a_\alpha)=1$.

Proof. (i) Let $x \in L, x \neq 0$ be such that $x \leqslant \operatorname{supp} m$. Then there is a state $m_1 \in P$ such that $m_1(x)=1$. From $m_1(\operatorname{supp} m)=1$ it follows that $L_m \subset L_{m_1}$, i.e., $m = m_1$. Hence, $x = \operatorname{supp} m$.

(ii) For $x \neq 0$ there is an $m \in P$ such that $m(x)=1$, then $\operatorname{supp} m \leqslant x$ and $\operatorname{supp} m$ is a point by (i).

(iii) Let $a \in L, a \neq 0$. For any $m \in P_a$, $\operatorname{supp} m \leqslant a$. Let $b \in L$ be such that $\operatorname{supp} m \leqslant b$ for any $m \in P_a$. Then $m(b)=1$ for any $m \in P_a$ imply that $P_a \subset P_b$ and from this it follows that $a \leqslant b$. Hence, $a = \bigvee\{\operatorname{supp} m: m \in P_a\}$. Now let $q \leqslant a$ be a point. Then there is an $m \in P$ such that $m(q)=1$. From $\operatorname{supp} m \leqslant q$ it follows that $\operatorname{supp} m = q$ and $q \leqslant a$ implies that $m \in P_a$. Hence, $a = \bigvee\{q: q \leqslant a, q \text{ is a point}\}$.

(iv) $m(a_\alpha)=1, \alpha \in A$ imply that $\operatorname{supp} m \leqslant a_\alpha, \alpha \in A$, i.e., $\operatorname{supp} m \leqslant \bigwedge \{a_\alpha: \alpha \in A\}$. Hence, $m(\wedge a_\alpha)=1$. ■

Points $e_1, e_2 \in L$ are perspective if there is a point $e_3 \in L, e_3 \neq e_1, e_2$ such that $e_3 \leqslant e_1 \vee e_2$.

Theorem 10. Let (L,M) be a quantum logic such that L is a complete lattice and all the states in P are supported. Then for any $m_1, m_2 \in P, m_1 \approx m_2$ if and only if $\operatorname{supp} m_1$ and $\operatorname{supp} m_2$ are perspective.

Proof. Let $m_1 \approx m_2$; then there is a state $m \in P$ such that $m \in \Lambda(\{m_1, m_2\})$, $m \neq m_1, m_2$. From this it follows that $\operatorname{supp} m_1 \leqslant a, \operatorname{supp} m_2 \leqslant a$ imply $\operatorname{supp} m \leqslant a (a \in L)$. Hence, $\operatorname{supp} m \leqslant \operatorname{supp} m_1 \vee \operatorname{supp} m_2$. Now let $\operatorname{supp} m_1$ and $\operatorname{supp} m_2$ be perspective. Let $e \in L$ be a point such that $e \leqslant \operatorname{supp} m_1 \vee \operatorname{supp} m_2, e \neq \operatorname{supp} m_1, \operatorname{supp} m_2$. Let $m \in P$ be such that $m(e) = 1$. Clearly, $\operatorname{supp} m = e$. If $a \in L$ is such that $m_1(a) = 1, m_2(a) = 1$, then $\operatorname{supp} m_1 \leqslant a, \operatorname{supp} m_2 \leqslant a$ imply $\operatorname{supp} m \leqslant a$, i.e., $m(a) = 1$. Hence, $m \in \Lambda(\{m_1, m_2\})$. ■

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